A NECESSARY AND SUFFICIENT CONDITION FOR THE BIBO STABILITY OF GENERAL-ORDER BODE-TYPE VARIABLE-AMPLITUDE WAVE-DIGITAL EQUALIZERS

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ABSTRACT

Recently, the authors developed a new synthesis technique for the design of higher-order Bode-type variable-amplitude (VA) wave-digital (WD) equalizers. The salient feature of the resulting VA WD equalizers is that they permit the continuous variation of the WD equalizer transfer function from a shaping transfer function to its inverse by changing the value of a single variable digital multiplier only. The proposed design technique was based on the WD realization of the corresponding positive-real analog prototype shaping impedance function, and on the realization of the equalizer transfer function as the reflectance of the shaping impedance function with respect to the constituent variable digital multiplier. This paper is concerned with an investigation of the bounded-input bounded-output (BIBO) stability of general-order VA WD equalizers. It is shown that the resulting conditions are both necessary and sufficient for the BIBO stability of the VA WD equalizers for the entire range of values for the variable digital multiplier. These conditions can be checked in a straightforward fashion in terms of the characteristics of the shaping transfer function alone. An application example is given to illustrate the main results.

1. INTRODUCTION

In his classical paper more than six decades ago, Bode [1] introduced the concept of variable-amplitude (VA) analog equalizers. These equalizers are capable of varying the gain associated with various frequency bands along the audio signal frequency spectrum, and find practical applications in multimedia, digital audio, digital signal enhancement/correction and hearing aids.

The magnitude-frequency response characteristic of a Bode-type VA analog equalizer satisfies a relationship of the form

\[ T_v(s) = \frac{r_v + T_s(s)}{1 + r_v T_s(s)} \]  

where \( T_v(s) \) is the transfer function of the VA equalizer, where \( T_s(s) \) is a fixed shaping transfer function, and where \( s = j\omega \) is the continuous-time (analog) complex frequency variable. Moreover, \( r_v = R_v/R_0 \), where \( R_v \) is a (positive) variable resistor, and where \( R_0 \) is a suitable reference resistance. In this way, the variation of \( r_v \) from 0 (via 1) to \( \infty \) results in a geometrical variation of \( |T_v(j\omega)| \) (via 1) to \( |T_s(j\omega)|^{-1} \). Equivalently, the variation of \( r_v \) from 0 (via 1) to \( \infty \) results in an arithmetical variation of \( |T_v(j\omega)|_{dB} \) from \( |T_s(j\omega)|_{dB} \) (via 0dB) to \( -|T_s(j\omega)|_{dB} \).

In a previous paper [2], the authors developed a synthesis technique for the design of higher-order Bode-type VA wave-digital (WD) [3] equalizers consisting of one single variable digital multiplier only. This design technique was based on the derivation of a corresponding positive-real analog prototype shaping impedance function, on the WD realization of the prototype shaping impedance function (employing the bilinear analog-to-digital frequency transformation), and on the realization of the desired equalizer transfer function as the reflectance of the shaping impedance function with respect to the constituent variable digital multiplier.

This paper is concerned with an investigation of the bounded-input bounded-output (BIBO) stability of general-order VA WD equalizers. These investigations are carried out in terms of the analog prototype equalizer transfer function by recognizing the fact that the bilinear analog-to-digital frequency transformation preserves the BIBO stability or instability in the corresponding VA WD equalizer.

2. THEORETICAL BACKGROUND

Let us consider a continuous-time analog prototype transfer function \( T_v(s) \) satisfying the following constraints:

**Constraint 1:** The analog prototype shaping transfer function \( T_v(s) \) is a real rational function of \( s \) (in order for \( T_v(s) \) to be realizable).

**Constraint 2:** \( a) 0 < |T_v(j\omega)| \leq 1 \) (used in what follows), or \( b) 1 \leq |T_v(j\omega)| < \infty \) for all \( \omega \) (in order for the sign of \( |T_v(j\omega)|_{dB} \) remain unchanged as \( r_v \) varies in the “half-interval” 0 to 1).

**Theorem 1:** In order for the analog prototype transfer function \( T_v(s) \) to be a BIBO stable transfer function for all values of the variable resistor \( r_v \), the shaping transfer function \( T_s(s) \) must be a minimum-phase function.

**Proof:** Since \( T_v(s) = T_s(s) \) for \( r_v = 0 \), and since \( T_v(s) = 1/T_s(s) \) for \( r_v = \infty \) (c.f. Eqn. 1), in order for \( T_v(s) \) to be a BIBO stable transfer function, it is necessary that both \( T_v(s) \) and \( 1/T_v(s) \) be BIBO stable transfer functions. This implies that the shaping transfer function \( T_v(s) \) must be a minimum-phase function.

**Theorem 2:** \( T_s(s) \) must be also a strictly minimum phase transfer function.

**Proof:** Since \( |T_v(j\omega)| \) and \( |1/T_v(j\omega)| \) are both bounded for all values of \( \omega \) (c.f. Constraint 2), \( T_s(s) \) must be devoid of both poles and zeros on the imaginary axis of the complex s-plane. Consequently, the minimum-phase transfer function \( T_v(s) \) (c.f. Theorem 1) must also be a strictly minimum-phase function.

1 Due to its symmetrical variation, sign of \( |T_v(j\omega)|_{dB} \) also remains unchanged when \( r_v \) varies in the other “half-interval”, i.e. from 1 to \( \infty \).
Let us assume that the shaping transfer function $T_s(s)$ is of general order $n$, having a magnitude-response characteristic which satisfies the high-level system design specifications

$$\quad q^{-1}|T_s(j\omega)| \leq 1 \quad \text{for} \quad \omega \in \Omega_p, \quad (2)$$
$$\quad h^{-1}|T_s(j\omega)| \leq p^{-1} \quad \text{for} \quad \omega \in \Omega_d, \quad (3)$$

where $\Omega_p (\Omega_d)$ represents the passband (stopband) frequency region(s) of the transfer function $T_1(s)$, and where $h > p > q > 1$ are equalizer design parameters. Moreover, let us define a pair of strictly BIBO stable rational continuous-time transfer functions $T_m(s)$ of orders $m_m$ in accordance with

$$\quad T_m(s) = \frac{N_m(s)}{D_m(s)} \quad \text{for} \quad m = 1, 2, \quad (4)$$

where $N_m(s)$ represents the numerator polynomial, and where $D_m(s)$ represents the denominator polynomial of $T_m(s)$. In addition, let the transfer functions $T_m(s)$ possess a pair of magnitude-response characteristics satisfying the system design specifications

$$\quad 0 \leq -|T_m(j\omega)|_{dB} \leq A_{pm} \quad \text{for} \quad \omega \in \Omega_{pm}, \quad (5)$$
$$\quad -|T_m(j\omega)|_{dB} \geq A_{am} \quad \text{for} \quad \omega \in \Omega_{am}, \quad (6)$$

in terms of passband ripples (stopband losses) $A_{pm}$ ($A_{am}$), and in terms of passband (stopband) frequency region(s) $\Omega_{pm}$ ($\Omega_{am}$). Finally, let the transfer functions $T_m(s)$ have the same orders $n_1 = n_2 = n$, and have the same passband (stopband) frequency region(s) $\Omega_{p1} = \Omega_{p2} = \Omega_d$ ($\Omega_{a1} = \Omega_{a2} = \Omega_p$). 

**Theorem 3:** The shaping transfer function $T_s(s)$ satisfying the above specifications can be derived as

$$\quad T_s(s) = \frac{1}{h} \frac{T_2(s)}{T_1(s)}, \quad (7)$$

provided that

$$\quad A_{p1} = 10 \log_p \frac{h^2 - 1}{p^2 - 1} \quad \text{and} \quad A_{a1} = 10 \log_p \frac{h^2 - 1}{q^2 - 1}, \quad (8)$$

and

$$\quad A_{p2} = 10 \log_p \frac{p^2 (h^2 - 1)}{h^2 (p^2 - 1)} \quad \text{and} \quad A_{a2} = 10 \log_p \frac{q^2 (h^2 - 1)}{h^2 (q^2 - 1)}. \quad (9)$$

**Proof:** See [2].

The transfer functions $T_m(s)$ will have the same numerator polynomials (to within multiplication by a constant) in accordance with

$$\quad N_2(s) = hN_1(s) \equiv N(s). \quad (10)$$

Therefore, from Eqns. 10, 7 and 4,

$$\quad T_s(s) = D_1(s)/D_2(s), \quad (11)$$

rendering $T_s(s)$ as a strictly minimum-phase transfer function (c.f. Theorem 2).

### 3. VA WD Equalizer Realization

Let us define a normalized driving-point impedance function $\hat{Z}(s)$ in accordance with

$$\quad \hat{Z}(s) = \frac{T_s(s) - 1}{T_s(s) + 1}, \quad (12)$$

**Theorem 4:** $\hat{Z}(s)$ in Eqn. 12 is a positive-real impedance function. 

**Proof:** According to Talbot [4], $\hat{Z}(s)$ is a positive-real impedance function since $T_s(s)$ has no poles in the right-half of the complex $s$-plane (c.f. Theorem 2), and since $|T_s(j\omega)|$ is bounded for all $\omega$ (c.f. Constraint 2).

In terms of Brune’s sections, the driving-point impedance function $\hat{Z}(s)$ can be realized as shown in Fig. 1, where the leftmost parallel tuned section realizes possible zeros at $s = 0$ and $s = \infty$, and where the remaining part realizes a minimum impedance function [4]. In this way, by using Eqn. 12, $T_s(s)$ can be expressed in terms of $\hat{Z}(s)$ in accordance with

$$\quad T_s(s) = \frac{1 + \hat{Z}(s)}{1 - \hat{Z}(s)}, \quad (13)$$

Then, by invoking Eqn. 13 in Eqn. 1, and by manipulating the resulting equation, one obtains

$$\quad T_v(s) = -\frac{\hat{Z}(s) - \hat{r}_{1v}}{\hat{Z}(s) + \hat{r}_{1v}}, \quad (14)$$

where

$$\quad \hat{r}_{1v} = \frac{1 + r_v}{1 - r_v} \quad (15)$$

Therefore, the transfer function $T_v(s)$ can be realized as the reflection of $\hat{Z}(s)$ with respect to $\hat{r}_{1v}$, leading to the desired VA WD equalizer as shown in Fig. 2. The resulting WD equalizer realizes a transfer function

$$\quad T_{v, WD}(z) = B_2(z)/E(z), \quad (16)$$

where $T_{v, WD}(z)$ is related to $T_v(s)$ through the bilinear analog-to-digital frequency transformation $s = 2f_s \frac{z - 1}{z + 1}$, where $z$ represents the discrete-time (digital) complex-frequency variable, and where $f_s$ represents the sampling frequency.

The VA WD realization in Fig. 2 consists (left to right) of a conventional parallel two-port adaptor [5] and a chain of two-port subnetworks, with each subnetwork being easily identified by using the corresponding realizations in Fig. 3 or their duals in Fig. 4 [6]. In this VA WD equalizer realization, the desired variations in $\hat{r}_{1v}$ can be implemented directly through the variation of the value of the single digital multiplier $m_v$ within the leftmost two-port parallel adapter [2].
4. BIBO STABILITY INVESTIGATION FOR THE RESULTING VA WD EQUALIZER

It is well known that the bilinear frequency transformation preserves the BIBO stability or instability of the analog prototype equalizer transfer function $T_v(s)$ in the resulting VA WD equalizer transfer function $T_{v, WD}(z)$. Therefore, without any loss in generality, the investigations of BIBO stability can be carried out in terms of the analog prototype equalizer transfer function $T_v(s)$.

The above BIBO stability was investigated in Theorems 1 and 2 for the extreme values $r_v = 0$ and $r_v = \infty$ of the variable resistor $r_v$. Therefore, it remains to investigate the BIBO stability for the intermediate values $0 < r_v < \infty$ of $r_v$. From Eqsns. 11 and 1,

$$T_v(s) = \frac{D_1(s) + r_vD_2(s)}{D_2(s) + r_vD_1(s)} \quad (17)$$

Therefor, the BIBO stability of $T_v(s)$ requires the examination of the roots of the characteristic equation

$$K(s) = D_2(s) + r_vD_1(s) = 0. \quad (18)$$

Let us express $T_v(s)$ in the general form

$$T_v(s) = \frac{d_{10} + d_{11}s^1 + \ldots + d_{1n}s^n}{d_{20} + d_{21}s^1 + \ldots + d_{2n}s^n}. \quad (19)$$

where the coefficients $d_{ni}$ (for $i = 1, 2, \ldots, n$) are (or can be made) non-negative².

**Theorem 5:** The Characteristic equation $K(s)$ in Eqn. 18 is degree invariant for all values of $0 < r_v < \infty$.

**Proof:** From Eqsns. 19 and 18, $K(s)$ will be degree-invariant provided that

$$d_{2n} + r_vd_{1n} \neq 0. \quad (20)$$

The proof is at once established by recognizing the fact that at least one of the two (non-negative) coefficients $d_{2n}$ or $d_{1n}$ is non-zero, and by recalling that $0 < r_v < \infty$.

**Theorem 6:** In order for $T_v(s)$ to be BIBO stable for all (intermediate) values $0 < r_v < \infty$ of the variable resistor $r_v$, it is sufficient for $T_v(s)$ to be a strictly minimum-phase transfer function.

**Proof:** If $K(s)$ is degree-invariant, then the roots of $K(s)$ will vary continuously along a constant number of root-locus branches [7]. The transfer function $T_v(s)$ will become BIBO unstable if and only if one or more of the root-locus branches associated with $K(s)$ intersects the imaginary axis of the complex $s$-plane, i.e. if

$$1 + r_vT_v(j\omega_0) = 0 \quad (21)$$

for one or more frequencies $\omega_0$. Since $r_v$ is a real-valued resistance, Eqn. 21 is equivalent to

$$\Re\{T_v(j\omega_0)\} = 0 \quad \text{and} \quad 1 + r_v\Re\{T_v(j\omega_0)\} = 0, \quad (22)$$

where $\Re\{\cdot\}$ and $\Im\{\cdot\}$ represent the real and imaginary parts of $\{\cdot\}$, respectively.

In this way, the transfer function $T_v(s)$ in Eqn. 17 will be BIBO stable for all values of $0 < r_v < \infty$ provided that,

$$\Re\{T_v(j\omega_0)\} = 0, \quad \text{then} \quad 1 + r_v\Re\{T_v(j\omega_0)\} > 0. \quad (23)$$

The real and imaginary parts of $T_v(j\omega)$ can be conveniently be obtained as

$$\Re\{T_v(j\omega)\} = \frac{\Re\{D_1(j\omega)\} \Re\{D_2(j\omega)\} + \Im\{D_1(j\omega)\} \Im\{D_2(j\omega)\}}{\Re\{D_2(j\omega)\}^2 + \Im\{D_2(j\omega)\}^2} \quad (24)$$

and

$$\Im\{T_v(j\omega)\} = -\frac{\Re\{D_1(j\omega)\} \Im\{D_2(j\omega)\} + \Im\{D_1(j\omega)\} \Re\{D_2(j\omega)\}}{\Re\{D_2(j\omega)\}^2 + \Im\{D_2(j\omega)\}^2} \quad (25)$$

From Eqn. 25, when $\Re\{T_v(j\omega_0)\} = 0$,

$$\Re\{D_1(j\omega_0)\} \Im\{D_2(j\omega_0)\} + \Im\{D_1(j\omega_0)\} \Re\{D_2(j\omega_0)\} = 0. \quad (26)$$

Subsequently, from Eqns. 26 and 24, when $\Re\{T_v(j\omega_0)\} = 0$,

$$\Re\{T_v(j\omega_0)\} = \Re\{D_1(j\omega_0)\} \Re\{D_2(j\omega_0)\} + \frac{\Re\{D_1(j\omega_0)\} \Re\{D_2(j\omega_0)\}}{\Re\{D_2(j\omega_0)\}^2 + \Im\{D_2(j\omega_0)\}^2}. \quad (27)$$

²This is due to the fact that $D_1(s)$ and $D_2(s)$ are strictly Hurwits polynomials.
But, $\Re\{D_1(j\omega)\} > 0$ and $\Re\{D_2(j\omega)\} > 0$ for all values of $\omega$ (since $D_1(s)$ and $D_2(s)$ are strictly Hurwitz polynomials). Consequently, when $\Im\{T_v(j\omega)\} = 0$, $\Re\{T_v(j\omega)\} > 0$, giving rise to satisfaction of Eqn. 23

**Theorem 7:** Subject to the satisfaction of Constraints 1 and 2, it is both necessary and sufficient for the analog prototype shaping transfer function $T_v(s)$ to be a strictly minimum-phase transfer function if it is required that the corresponding VA WD equalizer transfer function $T_vW_p(z)$ be BIBO stable for all values of the constituent variable digital multiplier $m_v$.

**Proof:** The proof is at once established by taking into account Theorems 3 and 6 together with the properties of the bilinear frequency transformation (as pointed out before).

### 5. APPLICATION EXAMPLE

Let us consider the design of a fourth-order bandpass elliptic VA WD equalizer satisfying the following specifications.

\[
\begin{align*}
  h &= 4.00 & p &= 3.5 & q &= 1.5 \\
  f_{p1} &= 8.00 \text{ Hz} & f_{p2} &= 10.00 \text{ Hz} \\
  f_{a1} &= 6.00 \text{ Hz} & f_{a2} &= 12.00 \text{ Hz} & 1/T &= 32.00 \text{ Hz}
\end{align*}
\]

Then, $A_{p1} = 1.249387366 \text{ dB}$, $A_{a1} = 28.753975887 \text{ dB}$, and $A_{p2} = -0.089548426 \text{ dB}$, $A_{a2} = 16.798711857 \text{ dB}$. Consequently,

\[
\begin{align*}
  d_{10} &= 0.99016629597150 \\
  d_{11} &= 30.78043908024169 \\
  d_{12} &= 13160.26562996701 \\
  d_{13} &= 188687.0835399217 \\
  d_{14} &= 37208559.89697220
\end{align*}
\]

Then, the variation of the magnitude-frequency response of the corresponding VA WD equalizer as a function of the variable resistor $r_v$ can be obtained as shown in Fig. 5. From Fig. 5, the VA WD equalizer magnitude-frequency response exhibits the desirable arithmetically symmetric variations (around 0 dB) for geometrically symmetric variations in $r_v$ values.

Finally, the root locus branches for the poles $p_i$ (for $i = 1, 2, 3, 4$) of the analog prototype equalizer transfer function $T_v(s)$ as a function of the variable resistor $r_v$ are as shown in Fig. 6. From Fig. 6, the poles of $T_v(s)$ remain in the left half of the complex $s$-plane, rendering $T_v(s)$ as BIBO stable for the entire range of values $0 \leq r_v \leq \infty$.

### 6. ACKNOWLEDGEMENTS

This work was supported, in part, by NSERC Operating Grant #A6715, and by Micronet.

### 7. REFERENCES


