Cramér-Rao Bounds for Estimation of Pure-Tone Signals’ Azimuth-Elevation Arrival Angles & Polarization Parameters & Frequencies Using a Dipole-Triad or a Loop-Triad

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Abstract—This work derives new non-asymptotic and asymptotic Cramér-Rao lower bounds (CRB) for the estimation of multiple pure-tone incident signals’ amplitudes, azimuth-elevation arrival-angles, polarization parameters, and frequencies, based on data observed from one dipole triad (composed of three spatially collocated but orthogonally oriented electrically short dipoles) or one loop-triad (composed of three spatially collocated but orthogonally oriented magnetically small loops). The incident sources are pure-tones at distinct frequencies, in contrast to the existing electromagnetic antenna-array signal-estimation CRB literature’s modeling of all sources to be at the same carrier-frequency.

Index Terms—Array Signal Processing, Blind Estimation, Direction of Arrival Estimation, Parameter Estimation

I. Introduction & Literature Overview

A. Electromagnetic Vector-Sensor Direction-Finding

A series of recent papers, [10], [11], [12], [15], [16], [17], [18] and [20] among others, investigate the direction-finding and/or polarization estimation performance obtainable from one electromagnetic “vector-sensor”, which comprises of a spatially collocated but diversely oriented collection of electrically short dipoles and/or magnetically small loops:

(A) a dipole-triad (consisting of three collocated but orthogonally oriented dipoles) collocated with a loop-triad (consisting of three collocated but orthogonally oriented dipoles) [10], [11], [12], [15], [17] and [20], or

(B) only the above-mentioned dipole-triad [18], or

(C) only the above-mentioned loop-triad [18], or

(D) the above dipole triad spatially displaced from the above loop triad [18], or

(E) two horizontally and orthogonally oriented loops collocated with a vertically oriented dipole [16], or

(F) two horizontally and orthogonally oriented loops [16].

Such an electromagnetic vector-sensor offers the following direction-finding advantages: (1) The polarization diversity among the vector-sensor’s component antennas allows incident sources to be separated on account of their polarization differences in addition to their azimuth/elevation angular differences. (2) The spatial collocation of all component-antennas in the vector-sensor means no spatial phase delay in the vector-sensor’s array manifold; hence, near-field sources may be located by an individual vector-sensor as well as far-field sources. (3) In a multi-source scenario, each source’s three Cartesian direction-cosine estimates (and thus each source’s azimuth-angle estimate and the elevation-angle estimate) are automatically paired without further post-processing.

B. Cramér-Rao Bound for Electromagnetic Vector-Sensor Direction-Finding & Polarization Estimation

Parameter-estimation errors’ lower bounds for direction-finding and/or polarization estimation using the collocated six-component electromagnetic vector-sensors of (3) have been defined and derived in [2], [7] and [10] for incident signals assumed to share a common carrier-frequency — these signals’ equivalent baseband representation are modeled as temporally uncorrelated complex-valued Gaussian stochastic processes. This signal model would be inapplicable to the new algorithms of [12] and [17]2, which estimate the arrival angles and polarization parameters of multiple pure-tone signals. Instead, these pure-tone signals need be modeled as deterministic sinusoids with: (a) deterministic, distinct but unknown frequencies, (b) deterministic but unknown amplitudes, arrival angles, polarization parameters, and (c) random and statistically uncorrelated temporal phases.

This work derives new Cramér-Rao bounds for a dipole-triad or a loop-triad vector-sensor, with the simplifying assumption of deterministic unknown temporal phases (instead of stochastic uniformly distributed uncorrelated phases as in (c) above). The will render the ensuing

2Although the Cramér-Rao bound in [18] was calculated with the incident-signal modeled as a temporally white complex-valued Gaussian process, equations (4-7), (9-12), (16-21) in [18] may modify [12] and [17] for use with the electromagnetic vector-sensor constructions (B), (C), (D). Similarly, the equations in section 3.1 and 3.2 of [16] may modify [12] and [17] for use with the electromagnetic vector-sensor constructions (F) and (E), respectively.
mathematics to be more tractable. This work will follow the general analysis approach in [19], which has derived analogous results but for an underwater acoustical vector-hydrophone.

II. A Dipole-Triad’s & a Loop-Triad’s Array Manifolds

The $k$th unit-power completely polarized transverse electromagnetic wave, having traveled through a homogeneous isotropic medium, is characterized by the 6-component electromagnetic array-manifold (comprising of the electric-field vector $e_k$ and the magnetic-field vector $h_k$), expressible in Cartesian coordinates as [2]:

$$
\begin{pmatrix}
\cos \phi_k \cos \psi_k & -\sin \phi_k \\
\sin \phi_k \cos \psi_k & \cos \phi_k \\
-\sin \psi_k & 0 \\
-\sin \phi_k & -\cos \phi_k \cos \psi_k \\
\cos \phi_k & -\sin \phi_k \cos \psi_k \\
0 & \sin \psi_k
\end{pmatrix}
$$

$$\equiv \Theta(\psi_k, \phi_k)$$

where $0 \leq \psi_k < \pi/2$ denotes the signal’s elevation angle measured from the vertical z-axis, $0 \leq \phi_k < 2\pi$ symbolizes the azimuth angle measured from the positive x-axis, $0 \leq \gamma_k < \pi/2$ refers to the auxiliary polarization angle, and $-\pi \leq \eta_k < \pi$ represents the polarization phase difference. Note that $\Theta(\psi_k, \phi_k)$ depends only on the azimuth-elevation arrival-angles, whereas $\Theta(\gamma_k, \eta_k)$ depends only on the polarization parameters.

Because $h(\psi, \phi, \gamma, \eta) = h(\psi_k, \phi_k, \gamma_k, \eta_k)e^{j\eta}$, the array manifold properties of the magnetic-loop triad are dual to those of an electrical-dipole triad. The dipole-triad’s and the loop-triad’s $3 \times 1$ array manifolds may be compactly expressed as:

$$
a_k =
\begin{pmatrix}
\cos \phi_k \cos \psi_k & -\sin \phi_k \\
\sin \phi_k \cos \psi_k & \cos \phi_k \\
-\sin \psi_k & 0 \\
-\sin \phi_k & -\cos \phi_k \cos \psi_k \\
\cos \phi_k & -\sin \phi_k \cos \psi_k \\
0 & \sin \psi_k
\end{pmatrix}
\equiv \Theta(\psi_k, \phi_k)
$$

$$\equiv s_k$$

where the “flag” $d$ equals 1 for the dipole-triad and $-1$ for the loop-triad.

III. Statistical Model of Collected Data

With a total of $K$ impinging pure-tone completely polarized sources, plus additive complex-valued zero-mean (and possibly spatio-temporally correlated) noise at each constituent-antenna of the electromagnetic vector-sensor [12],

$$
\mathbf{z}(t_n) \equiv [a_1, \ldots, a_K] \\
\begin{bmatrix}
s(t_n, f_1) \\
\vdots \\
s(t_n, f_K)
\end{bmatrix} +
\begin{bmatrix}
n(t_n, f_1) \\
\vdots \\
n(t_n, f_K)
\end{bmatrix}
$$

$$s(t_n, f_k) \equiv b_k e^{j(2\pi f_k t_n + \varphi_k)} \quad ; k = 1, \ldots, K
$$

where $b_k$ denotes the $k$th source’s amplitude, $f_k$ symbolizes the $k$th signal’s frequency, $\varphi_k$ refers to the $k$th signal’s temporal phase. The above model of the incident signals differ from that in [2], [7], [10] and [18], which model the incident signals as a set of statistically independent and temporally uncorrelated zero-mean complex-Gaussian random sequences.

The entire set of data measurements is:

$$
\mathbf{Z} \equiv [\mathbf{z}(t_1) \ldots \mathbf{z}(t_N)]
$$

where $\{t_1, \ldots, t_N\}$ represents the set of distinct time-sampling instants.

The parameter-estimation problem, whose performance this work aims to lower-bound, is to determine the unknown deterministic scalars $\{b_k, f_k, \psi_k, \phi_k, \gamma_k, \eta_k, k = 1, \ldots, K\}$ from the data set $\mathbf{Z}$ above without any a priori knowledge of $\{\varphi_k, k = 1, \ldots, K\}$.

Define $\theta_k \equiv [\theta_k^T, \ldots, \theta_k^T]^T$, where $\theta_k \equiv [b_k, \psi_k, \phi_k, \gamma_k, \eta_k, \varphi_k, f_k]^T$. (Note that $\varphi_k$, herein defined as the $k$th signal’s phase in the middle of the given time interval $(1, N)$, differs from the $\varphi_k$ of (5).) Let $\mathbf{z} = \text{vec}(\mathbf{Z})$ be a vector consisting of all collected time samples; thus, $\mathbf{z} \sim \mathcal{N}(\mu(\theta, \Gamma), \Sigma)$, where $\mu(\theta, \Gamma) \equiv \sum_{k=1}^{K} b_k \mathbf{a}_k \otimes \mathbf{s}_k$, $\otimes$ denotes the Kronecker product, and $\mathbf{s}_k \equiv \mathbf{s}(f_k, \varphi_k)$

$$
\equiv [e^{j[2\pi f_k (1 - \frac{\delta}{2} - \frac{\delta}{2}) + \varphi_k]}, \ldots, e^{j[2\pi f_k (N - \frac{N - \delta}{2}) + \varphi_k]}]^T
$$

All noise samples $\{n_k(t_n), k = 1, \ldots, 4, n = 1, \ldots, N\}$, are herein modeled as zero-mean, circular complex Gaussian with known spatio-temporal covariance matrix $\Gamma$ (of size $3L \times 3L$) drawing its elements from the spatio-temporal covariance function

$$
c(k, l, m, n = 1, k = 1, \ldots, K, n = 1, \ldots, N) = \mathbb{E}[n^*_k(t_n) n_l(t_m)]
$$

The subsequent exposition will assume temporally uncorrelated and invariant spatio-polarizational covariance matrix $\Gamma = \text{diag}(\sigma^2_{x_k}, \sigma^2_{y_k}, \sigma^2_{z_k})$, where $\sigma^2_{x_k}$ is noise variance at each dipole in the dipole-triad. For the loop-triad, $\Gamma = \text{diag}(\sigma^2_{x_k}, \sigma^2_{y_k}, \sigma^2_{z_k})$. Hence,

$$
\Gamma = \Gamma_0 \otimes \mathbf{I}_{N \times N}
$$
IV. Derivation of New Cramér-Rao Bounds

The Cramér-Rao bound analysis in [2], [7], [10] and [18] models the set of all incident signals as a set of statistically independent temporally and uncorrelated zero-mean complex-Gaussian random sequences. That model is inapplicable to the present scheme where the set of incident signals are uncorrelated pure sinusoids. Though Cramér-Rao bound analysis has been available in [1], [3], [4], [5], [6], [8], [9] and [13 ] for multiple pure-tone signals, those work do not directly account for the present “vector-sensor” array manifolds.

A. Constructing the Fisher Information Matrix

The $(7K \times 7K)$-size Fisher information matrix equals:

$$
J(\theta) = 2 \text{Re} \left[ \left( \frac{\partial \mu(\theta)}{\partial \theta} \right)^H \Gamma_{11}^{-1} \frac{\partial \mu(\theta)}{\partial \theta} \right] \tag{10}
$$

where elements of the $(3N \times 7K)$ matrix $\partial \mu(\theta)/\partial \theta$ are:

$$
\frac{\partial \mu(\theta)}{\partial b_k} = a_k \otimes s_k \tag{11}
$$

$$
\frac{\partial \mu(\theta)}{\partial \psi_k} = b_k \frac{\partial \Theta_k}{\partial \psi_k} \otimes s_k = b_k \left( \frac{\partial \Theta_k}{\partial \psi_k} g_k \right) \otimes s_k \tag{12}
$$

$$
\frac{\partial \mu(\theta)}{\partial \phi_k} = b_k \frac{\partial \Theta_k}{\partial \phi_k} \otimes s_k = b_k \left( \frac{\partial \Theta_k}{\partial \phi_k} g_k \right) \otimes s_k \tag{13}
$$

$$
\frac{\partial \mu(\theta)}{\partial \gamma_k} = b_k \frac{\partial \Theta_k}{\partial \gamma_k} \otimes s_k = b_k \left( \Theta_k \frac{\partial g_k}{\partial \gamma_k} \right) \otimes s_k \tag{14}
$$

$$
\frac{\partial \mu(\theta)}{\partial \eta_k} = b_k \frac{\partial \Theta_k}{\partial \eta_k} \otimes s_k = b_k \left( \Theta_k \frac{\partial g_k}{\partial \eta_k} \right) \otimes s_k \tag{15}
$$

$$
\frac{\partial \mu(\theta)}{\partial \varphi_k} = j b_k a_k \otimes s_k \tag{16}
$$

$$
\frac{\partial \mu(\theta)}{\partial f_k} = 2 \pi b_k a_k \otimes \hat{s}_k \tag{17}
$$

with $\hat{s}_k \overset{\text{def}}{=} j \left[ 1 - \frac{N+1}{2}, 2 - \frac{N+1}{2}, \ldots, N - \frac{N+1}{2} \right]^T \otimes s_k$, and $\otimes$ represents the element-wise (Hadamard) product operator.

B. Simplifying the Fisher Information Matrix

Representing the $(7K \times 7K)$-size $J(\theta)$ in a block-matrix form:

$$
J(\theta) = \begin{bmatrix}
J_{1,1} & \ldots & J_{1,K} \\
\vdots & & \vdots \\
J_{K,1} & \ldots & J_{K,K}
\end{bmatrix} \tag{18}
$$

the $(k,t)$th $(7 \times 7)$-size block refers to:

$$
\begin{bmatrix}
J_{b_k,b_t} & J_{b_k,\psi_t} & J_{b_k,\phi_t} & J_{b_k,\gamma_t} & J_{b_k,\eta_t} & J_{b_k,\varphi_t} & J_{b_k,f_t} \\
J_{\psi_k,b_t} & J_{\psi_k,\psi_t} & J_{\psi_k,\phi_t} & J_{\psi_k,\gamma_t} & J_{\psi_k,\eta_t} & J_{\psi_k,\varphi_t} & J_{\psi_k,f_t} \\
J_{\phi_k,b_t} & J_{\phi_k,\psi_t} & J_{\phi_k,\phi_t} & J_{\phi_k,\gamma_t} & J_{\phi_k,\eta_t} & J_{\phi_k,\varphi_t} & J_{\phi_k,f_t} \\
J_{\gamma_k,b_t} & J_{\gamma_k,\psi_t} & J_{\gamma_k,\phi_t} & J_{\gamma_k,\gamma_t} & J_{\gamma_k,\eta_t} & J_{\gamma_k,\varphi_t} & J_{\gamma_k,f_t} \\
J_{\eta_k,b_t} & J_{\eta_k,\psi_t} & J_{\eta_k,\phi_t} & J_{\eta_k,\gamma_t} & J_{\eta_k,\eta_t} & J_{\eta_k,\varphi_t} & J_{\eta_k,f_t} \\
J_{\varphi_k,b_t} & J_{\varphi_k,\psi_t} & J_{\varphi_k,\phi_t} & J_{\varphi_k,\gamma_t} & J_{\varphi_k,\eta_t} & J_{\varphi_k,\varphi_t} & J_{\varphi_k,f_t} \\
J_{f_k,b_t} & J_{f_k,\psi_t} & J_{f_k,\phi_t} & J_{f_k,\gamma_t} & J_{f_k,\eta_t} & J_{f_k,\varphi_t} & J_{f_k,f_t}
\end{bmatrix}
$$

For arbitrary but size-compatible vectors $a, b, e, f$ and matrices $B, C$, it holds that $(a \otimes b)^H (B \otimes C)(e \otimes f) = a^H B b^H C f$. Hence, using (9) and (10-17),

$$
\begin{align*}
J_{b_k,b_t} &= \frac{2}{\sigma_e^2} a_k^T \Gamma_{00}^{-1} a_t \text{Re} \left[ s_k^H M_p^{-1} s_t \right] \tag{19} \\
J_{b_k,\psi_t} &= \frac{2}{\sigma_e^2} b_k a_t^T \Gamma_{00}^{-1} \frac{\partial a_t}{\partial \psi_t} \text{Re} \left[ s_k^H M_p^{-1} s_t \right], \tag{20}
\end{align*}
$$

$$
... k, t = 1, \ldots, K.
$$

The asymptotic behavior of $\{J_{k,t}, k, t = 1, \ldots, K\}$ will next be studied, assuming $N$ to be large and the signal frequencies to be fixed and distinct. The asymptotic behaviors of the above mentioned matrices are determined by the products $s_k^H M_p^{-1} s_t$ and $s_k^H M_p^{-1} s_t$. The off-diagonal elements of these matrices (corresponding to two different incident sources), after appropriate normalization as shown below, are asymptotically negligible [19]. The definition of $\{s_k, k = 1, \ldots, K\}$ gives (21), where $\Delta_{k,t} = \pi (f_k - f_t)$. As a result, for large $N, k \neq t$, and $k, t = 1, \ldots, K, J_{k,t} =$

$$
\begin{bmatrix}
O(1) & O(1) & O(1) & O(1) & O(1) & O(1) & O(N) \\
O(1) & O(1) & O(1) & O(1) & O(1) & O(1) & O(N) \\
O(1) & O(1) & O(1) & O(1) & O(1) & O(1) & O(N) \\
O(1) & O(1) & O(1) & O(1) & O(1) & O(1) & O(N) \\
O(1) & O(1) & O(1) & O(1) & O(1) & O(1) & O(N) \\
O(1) & O(1) & O(1) & O(1) & O(1) & O(1) & O(N) \\
O(1) & O(1) & O(1) & O(1) & O(1) & O(1) & O(N^2)
\end{bmatrix}
$$

(22)

and for $k = 1, \ldots, K$, (using some relationships concerning the array-manifolds in (3) to be presented in the next subsection) $J_{k,k}$:

$$
\begin{bmatrix}
O(N) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & O(N) & O(N) & 0 & 0 & 0 & 0 \\
0 & O(N) & O(N) & O(N) & O(N) & 0 & 0 \\
0 & 0 & O(N) & O(N) & 0 & 0 & 0 \\
0 & 0 & O(N) & O(N) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & O(N^3)
\end{bmatrix}
$$

(23)

Define the $7 \times 7$ normalizing matrix $D_N = \text{diag}(N^{1/2}, N^{1/2}, N^{1/2}, N^{1/2}, N^{1/2}, N^{3/2}, N^{3/2})$ and the $7K \times 7K$ normalizing matrix $D_N = \text{diag}(D_N, \ldots, D_N)$. All elements in the matrices $\{D_N^{-1} J_{k,t} D_N^{-1}, \text{all } k \neq t\}$ are asymptotically $O(1)$, while other elements have the order $O(1/N)$. Hence, the matrix $D_N^{-1} J(\theta) D_N^{-1}$ is asymptotically block diagonal. The diagonal of $D_N^{-1} J(\theta) D_N^{-1}$ consists of the elements $N^{-1} J_{b_k,b_k}, N^{-1} J_{f_k,f_k}$ and the blocks $N^{-1} J_{\psi_k,\psi_k} \Gamma_0^{-1} \{\eta_k, \eta_k\}^T$ for $k = 1, \ldots, K$.

This above analysis implies that each incident source’s parameters’ Cramér-Rao bounds may be analyzed without consideration of the other sources’ presence. This above analysis also means that the parameters $b_k$ and $f_k$ are each asymptotically decoupled from the 5-tuples $(\psi_k, \phi_k, \gamma_k, \eta_k, \varphi_k)$ for $k = 1, \ldots, K$. The Cramér-Rao bounds for these parameters, defined as proper diagonal elements and diagonal blocks of $J$, approximately equal the inverse of

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4The next two subsections’ derivation parallels that in [19].
\[ J_{b_k,b_k}, J_{f_k,f_k} \text{ and } J(\psi_k, \phi_k, \eta_k, \gamma_k, \varphi_k), \] respectively. The approximate CRB expression for \( f_k \) is proportional to \( N^{-3} \), as is usual in frequency estimation; and the Cramér-Rao bounds for the other parameters (including the angles of arrival) is proportional to \( N^{-1} \).

C. Explicit Expressions of the Cramér-Rao Bounds

Towards deriving explicit expressions of the incident signals' parameters' Cramér-Rao bounds, the following relations exist with respect to the dipole-triad’s and the loop-triad’s array-manifolds in (3) for all \( k \):

\[ \Theta_k^H \Theta_k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

\[ g_k^H g_k = 1 \]

For any scalar constants \( A \) and \( B \),

\[ g_k^H \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} g_k = A \sin^2(\gamma_k - \pi/4 + d\pi/4) + B \cos^2(\gamma_k - \pi/4 + d\pi/4) \]

which is entirely real-valued.

\[ \left( \begin{array}{c} \sin \gamma_k - \pi/4 + d\pi/4 \\ \cos \gamma_k - \pi/4 + d\pi/4 \end{array} \right) e^{-j\eta_k} + B \sin(\gamma_k - \pi/4 + d\pi/4) e^{-j\eta_k} + B \sin(\gamma_k - \pi/4 + d\pi/4) e^{-j\eta_k} + B \sin(\gamma_k - \pi/4 + d\pi/4) e^{-j\eta_k} \]

(28)

\[ = (A-B) B \sin(\gamma_k - \pi/4 + d\pi/4) \]

\[ \cos(\gamma_k - \pi/4 + d\pi/4)(e^{\eta_k} - e^{-j\eta_k}) \]

\[ = 2B \sin(\gamma_k - \pi/4 + d\pi/4) \sin \eta_k \]

(29)

The last expression is entirely imaginary-valued.

Furthermore,

\[ \frac{\partial g_k}{\partial \gamma_k} = \left[ \begin{array}{c} \cos(\gamma_k - \pi/4 + d\pi/4) e^{j(1+d)\eta_k/2} \\ -\sin(\gamma_k - \pi/4 + d\pi/4) e^{j(1-d)\eta_k/2} \end{array} \right] \]

\[ \frac{\partial g_k}{\partial \eta_k} = \left[ \begin{array}{c} j\frac{1+d}{2} \sin(\gamma_k - \pi/4 + d\pi/4) e^{j(1+d)\eta_k/2} \\ j\frac{1-d}{2} \cos(\gamma_k - \pi/4 + d\pi/4) e^{j(1-d)\eta_k/2} \end{array} \right] \]

\[ g_k^H \frac{\partial g_k}{\partial \gamma_k} = 0 \]

\[ g_k^H \frac{\partial g_k}{\partial \eta_k} = j(1+d)/2 \sin^2(\gamma_k - \pi/4 + d\pi/4) + j(1-d)/2 \cos^2(\gamma_k - \pi/4 + d\pi/4) \]

(30)

For the \((k,k)\)-th diagonal block-matrix's diagonal elements,

\[ \Gamma_k^{-1} = \frac{1}{\sigma^2} \]

\[ \frac{\partial \Theta_k}{\partial \psi_k} = \left[ \begin{array}{c} -\cos \phi_k \sin \psi_k \\ -\sin \phi_k \sin \psi_k \\ -\cos \psi_k \end{array} \right] \]

\[ \frac{\partial \Theta_k}{\partial \phi_k} = \left[ \begin{array}{c} -\sin \phi_k \cos \psi_k \\ -\cos \phi_k \cos \psi_k \\ -\sin \phi_k \end{array} \right] \]

\[ \Theta_k^T \frac{\partial \Theta}{\partial \psi} = \left[ \begin{array}{c} 0 \\ -\cos \psi_k \end{array} \right] \]

\[ \Theta_k^T \frac{\partial \Theta}{\partial \phi} = \left[ \begin{array}{c} 0 \\ -\cos \psi_k \end{array} \right] \]

For the loop-triad, the diagonal elements of the non-asymptotic information matrix \( J(\theta) \) can be shown to take on these values:

\[ J_{b_k,b_k} = \frac{2N}{\sigma^2} \]

\[ J_{\psi_k,\psi_k} = \frac{2b_k^2 N}{\sigma^2} \sin^2 \gamma_k \]

\[ J_{\phi_k,\phi_k} = \frac{2b_k^2 N}{\sigma^2} \cos^2 \gamma_k \sin^2 \gamma_k + \cos^2 \gamma_k \]

\[ J_{\gamma_k,\gamma_k} = \frac{2b_k^2 N}{\sigma^2} \]

\[ J_{\eta_k,\eta_k} = \frac{2b_k^2 N \sin^2 \gamma_k}{\sigma^2} \]
\[ J_{e,b_k} = \frac{2\beta_e^2 N}{\sigma_e^2} \quad \text{(43)} \]
\[ J_{h,f_k} = \frac{2\pi^2 b_h^2 N(N^2-1)}{3\sigma_h^2} \quad \text{(44)} \]

Furthermore, the signal parameters’ asymptotic Cramér-Rao bounds (for large \( N \) and well separated frequencies, \( 2\pi|f_k - f_i| \gg \frac{1}{N} \)) are inversely related to the above elements in (38) to (44). Inverting the \( 7 \times 7 \) block-matrix yields the following asymptotic Cramér-Rao bounds:

\[ \text{CRB}_{e,b_k} = \frac{\sigma_e^2}{2N} \]
\[ \text{CRB}_{e,\psi_k} = \frac{\sigma_e^2}{2Nb_k^2 \sin^2 \gamma_k \sin^2 \eta_k} \]
\[ \text{CRB}_{e,\phi_k} = \frac{\sigma_e^2}{2Nb_k^2 \cos^2 \gamma_k \sin^2 \eta_k \sin^2 \psi_k} \]
\[ \text{CRB}_{e,\gamma_k} = \frac{\sigma_e^2 (\cos^2 \psi_k \cos^2 \gamma_k + \sin^2 \psi_k \cos^2 \gamma_k \sin^2 \eta_k)}{2Nb_k^2 \cos^2 \gamma_k \sin^2 \eta_k \sin^2 \psi_k} \]
\[ \text{CRB}_{e,\eta_k} = \frac{\sigma_e^2 (\cos^2 \psi_k \cos^2 \gamma_k \sin^2 \eta_k + \sin^2 \psi_k \cos^2 \gamma_k \sin^2 \eta_k)}{2Nb_k^2 \sin^2 \gamma_k \cos^4 \gamma_k \sin^2 \psi_k} \]
\[ \text{CRB}_{e,\rho_k} = \frac{3\sigma_e^2}{2\pi^2 b_h^2 N(N^2-1)} \]

The corresponding asymptotic Cramér-Rao bounds for the loop-triad may be shown to equal:

\[ \text{CRB}_{h,b_k} = \frac{\sigma_h^2}{2N} \]
\[ \text{CRB}_{h,\psi_k} = \frac{\sigma_h^2}{2Nb_h^2 \cos^2 \gamma_k \sin^2 \eta_k} \]
\[ \text{CRB}_{h,\phi_k} = \frac{\sigma_h^2}{2Nb_h^2 \sin^2 \gamma_k \sin^2 \psi_k \sin^2 \eta_k} \]
\[ \text{CRB}_{h,\gamma_k} = \frac{\sigma_h^2 (\cos^2 \psi_k \cos^2 \gamma_k + \sin^2 \psi_k \sin^2 \gamma_k \sin^2 \eta_k)}{2Nb_h^2 \sin^2 \gamma_k \sin^2 \psi_k \sin^2 \eta_k} \]
\[ \text{CRB}_{h,\eta_k} = \frac{\sigma_h^2 (\cos^2 \psi_k \cos^2 \gamma_k \sin^2 \eta_k + \sin^2 \gamma_k \sin^2 \psi_k - 4 \cos^2 \gamma_k \cos^2 \psi_k \sin^2 \gamma_k)}{2Nb_h^2 \sin^2 \gamma_k \sin^2 \psi_k} \]
\[ \text{CRB}_{h,f_k} = \frac{3\sigma_h^2}{2\pi^2 b_h^2 N(N^2-1)} \]

Comparing the dipole-triad’s Cramér-Rao bounds with the loop-triad’s Cramér-Rao bounds, the former are related to the latter by substituting \( \cos^2 \gamma_k \) with \( \sin^2 \gamma_k \), \( \sin^2 \gamma_k \) with \( \cos^2 \gamma_k \), and \( \psi_k \) with \( \eta_k \).

The Cramér-Rao bounds are minimized when the polarization phase difference \( \eta_k \) or the elevation angle \( \psi_k \) at \( \pi/2 \) radians.

Furthermore, the azimuth arrival angle \( \phi_k \) does not influence the Cramér-Rao bounds.

References