RATIONAL SIGNAL SUBSPACE APPROXIMATION
WITH APPLICATIONS TO DOA ESTIMATION

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ABSTRACT

In this paper, we have developed an approach for approximating the signal and noise subspaces which avoid the costly eigendecomposition or SVD. These subspaces were approximated using rational and power-law methods applied to the sample covariance matrix. It is shown that MUSIC and Minimum Norm frequency estimators can be derived using these approximated subspaces. These approximate estimators are shown to be robust against noise and overestimation of number of sources. A substantial computational saving would be gained compared with those associated with the eigendecomposition-based methods. Simulations results show that these approximated estimators have comparable performance at low signal-to-noise ratio (SNR) to their standard counterparts and are robust against overestimating the number of impinging signals.

1. INTRODUCTION

Estimation of sinusoidal frequencies and direction of arrival (DOA) embedded in white noise is a problem of interest in many fields of signal processing such as geophysics, radar, and sonar. Several high resolution techniques emerged during the years for solving this problem. For useful articles and books the reader is referred to [2], [5]-[9] and the references therein. Several subspace approximation techniques were proposed in the literature [2], [6]-[9]. In this paper, we proposed an invariant subspace decomposition based on rational approximation.

The DOA can be described as follows. Consider a linear array of p sensors and q multiple narrowband signals impinging on the array with DOA angles $\theta_1, \theta_2, \ldots, \theta_q$. The received signal at the array can be expressed as

$$y(k) = A(\theta)s(k) + v(k),$$

where $s(k) \in \mathbb{C}^q$ ($\mathbb{C}$ is the field of complex numbers) is a vector of complex signals of q wavefronts $s(k) = [s_1(k), s_2(k), \ldots, s_q(k)]^T$, $v(k)$ is a $p \times 1$ vector of additive noise in sensors with $v(k) = [v_1(k), v_2(k), \ldots, v_p(k)]^T$ and $A$ is $p \times q$ matrix $A(\theta) = [a(\theta_1), a(\theta_2), \ldots, a(\theta_q)]$ with $a(\theta)$ being the steering vector of the array toward the direction $\theta$. It is also assumed that the signals and additive noise are stationary and zero-mean ergodic complex-valued random processes such that $E[v_i(k)v^*_j(k)] = \sigma^2_n \delta(i-j)$ for $i,j = 1, \ldots, q$. Here $E$, * denote the expectation, and conjugate transpose operators, respectively. Thus the spatial $p \times p$ covariance matrix of the array output is given by $R_y = E[yy^*(k)] = A(\theta)R_sA(\theta)^* + \sigma^2_nI_p$ with $R_s = E[ss^*(k)]$ is $q \times q$ matrix of amplitudes of $s$ and $I_p$ is the $p \times p$ identity matrix. If the $\theta$'s are all distinct, the unknown matrix $A \in \mathbb{C}^{p \times q}$ is of rank $q$. The main objective is to estimate $q$ and $A$ from the noisy data $\{y(k)\}_{k=1}^N$. In the next theorem we present a rational approximation of the eigenspace of the sample covariance matrix, $\hat{R}_y$. Theorem 1. Let $\hat{R}_y = \sum_{i=1}^p \lambda_i u_i u_i^*$, where $\lambda_i$ and $u_i$ be the $i$th eigenvalue and corresponding eigenvector. Assume that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_q > \lambda_{q+1} = \ldots = \lambda_p = \sigma^2_n$ and $u_i u_i^* = \delta_{i-j}$. Let $\{b_i\}_{i=1}^r$ be a set of real number such that $0 < b_1 < b_2 < \ldots < b_r$, and set $U_0 = \sum_{i,j} \delta_{i,j} u_i u_j^*$, $U_i = \sum_{i,j} \delta_{i,j} u_i u_j^*$ for $i = 1, \ldots, r$. Then

$$\lim_{m \to \infty} \prod_{i=1}^r (b_i I_p - \hat{R}_y)(b_i I_p + \hat{R}_y)^{-1} = \sum_{i=0}^r (-1)^i U_i.$$ (2)

Proof: See [6].

In Theorem 1 if $r = 1$ and $b_1$ is a threshold that separates the signal and noise eigenvalues, then a rational approximation of the signal and noise subspaces can be obtained as in the following result.

Corollary 2. Let $\hat{R}_y = \sum_{i=1}^p \lambda_i u_i u_i^*$ such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_q > \lambda_{q+1} = \ldots = \lambda_p = \sigma^2_n$ and $u_i u_i^* = \delta_{i-j}$, where $\lambda_i$ and $u_i$ are the $i$th eigenvalue and corresponding eigenvector. Let $b > 0$ be such that $\lambda_q > b > \lambda_{q+1}$ and let $U_s = \sum_{i=1}^q u_i u_i^*$ and $U_n = \sum_{i=q+1}^p u_i u_i^*$, then...
(i) \((b^m I_p - \hat{R}_y^m)(b^m I_p + \hat{R}_y^m)^{-1}\) converges to \(U := U_n - U_s\) (as \(m \to \infty\)), and therefore \(U_s = \frac{I_p - U}{2}\) and \(U_n = \frac{I_p + U}{2}\).

(ii) \(\hat{R}_y^m(b^m I_p + \hat{R}_y^m)^{-1}\) converges to \(U_s\) (as \(m \to \infty\)).

(iii) \(b^m(b^m I_p + \hat{R}_y^m)^{-1}\) converges to \(U_n\) (as \(m \to \infty\)).

**Proof:** follows from Theorem 1 by setting \(r - 1\).

### 2. HIGH RESOLUTION ESTIMATORS

Given the projections \(U_s\) and \(U_n\) onto the signal and noise subspaces, the exact MUSIC and minimum norm estimators are given by

\[
P_{\text{MUSIC}}(\theta) = \frac{1}{|a^*(\theta)U_n a(\theta)|^2},
\]

and

\[
P_{\text{MN}}(\theta) = \frac{1}{|a^*(\theta)U_n e_1|^2},
\]

where \(e_1\) is the first column of the \(p \times p\) identity matrix.

Let \(b\) be a number that separates the signal and noise eigenvalues, i.e., \(\lambda_i < b < \lambda_j\) for \(i = 1, \ldots, q\) and \(j = q + 1, \ldots, p\). Then an approximated MUSIC and Minimum-Norm estimators based on (ii) of Corollary 2 can be written as

\[
P_{b,\text{MUSIC}}^{(m)}(\theta) = \frac{1}{|p - a^*(\theta)\hat{R}_y^m(b^m I_p + \hat{R}_y^m)^{-1}a(\theta)|},
\]

and

\[
P_{b,\text{MN}}^{(m)}(\theta) = \frac{1}{|a^*(\theta)(I_p - \hat{R}_y^m(b^m I_p + \hat{R}_y^m)^{-1})e_1|^2}.
\]

A rough estimate of a threshold \(b\) is \(\frac{Tr(\hat{R}_y)}{p}\). This idea along with part (i) of Corollary 2 are incorporated in the following algorithms. Here we used \(Tr(A)\) to mean the trace of \(A\).

**Algorithm (Rational-MUSIC and Rational-Min-Norm)**

(i) Compute \(\hat{R}_y(l) = [r_y(j + l, k + l)]\), where \(r_y(j, k) = E[y(m + j)y^*(m + k)]\).

(ii) Choose \(m \geq 1, l \geq p\) and compute

\[
F^{(l)}_m = \left(\frac{Tr(\hat{R}_y(l))}{p}\right)^m I_p - \hat{R}_y(l)^m
\]

\[
F_{m}^{(l)} = \left(\frac{Tr(\hat{R}_y(l))}{p}\right)^m I_p + \hat{R}_y^m(l)^{-1}
\]

and let \(\bar{F}_m^{(l)} = \frac{F_m^{(l)} + F_{m}^{(l)*}}{2}\).

(iii) Compute the approximated noise subspace \(\hat{U}_n = \frac{I_p + F_m^{(l)}}{2}\).

(iv) Compute \(P_{b,\text{MUSIC}}^{(m)}(\theta)\) and \(P_{b,\text{MN}}^{(m)}(\theta)\) using

\[
P_{b,\text{MUSIC}}^{(m)}(\theta) = \frac{1}{|a^*(\theta)\hat{U}_n a(\theta)|},
\]

\[
P_{b,\text{MN}}^{(m)}(\theta) = \frac{1}{|a^*(\theta)\hat{U}_n e_1|},
\]

and locate the peaks. The frequencies are estimated as the angular positions of the peaks.

### 3. OPERATION COUNT

The methods presented in the previous sections are multiplication rich in that for a given \(m\), \(\hat{R}_y^m\) is required and followed by one matrix inversion operation. Matrix multiplication can be obtained very efficiently applying the Strassen algorithm [4]. In this algorithm, if \(A \in \mathbb{R}^{p \times p}\) and \(B \in \mathbb{R}^{p \times p}\) with \(p\) is a power of 2, then \(C = AB\) can be obtained with \(s \approx 2.807\) multiplications. Thus asymptotically, the number of multiplications in the Strassen algorithm is \(O(p^2 \cdot 2^{\log_2 7})\) compared with \(O(p^3)\) in the conventional method. It should be mentioned that in [1], Bailey implemented a Strassen approach that required only 60\% of the time needed by the conventional multiplication.

The number of flops in computing \((b^m I_p - \hat{R}_y^m)^{-1}(b^m I_p + \hat{R}_y^m)\) consists of approximately the number of flops in computing \(\hat{R}_y^m\) and the matrix inverse. Assuming that \(m = 2^r\), both of these processes cost about \(rp^{2.807} + \frac{p^3}{3}\) multiplications. The number of flops required to compute the SVD of \(\hat{R}_y\) by the Golub-Reinsch algorithm is \(21p^3\) [3]. For example, if we choose \(r\) to be 4 which correspond to \(m = 16\), a value that is very high in most applications, the number of flops required in the rational MUSIC is \(4p^2.807 + \frac{p^3}{3}\) which is still much less than \(21p^3\) using the Golub-Reinsch Algorithm [3].

Efficient matrix inversion can be computed using the LU decomposition. Once the LU factorization of \(A\) is known, the inverse matrix \(A^{-1}\) can be computed in \(\frac{(p-1)(2p-1)}{3}\) flops [3]-[4]. Thus the total number of flops involved in computing \((b^m I_p - \hat{R}_y^m)^{-1}(b^m I_p + \hat{R}_y^m)\) is about \(4p^3 + 2rp^3 = (2r + 1.333)p^3\). This number is still far less than the flop count for computing the SVD, which is about \(21p^3\), for \(r < 10\). Note that \(r = 9\) corresponds to \(m = 512\) which is extremely large for most applications. Thus for all practical purposes these algorithms which are based on Corollary 2 are less costly than the truncated SVD-based methods.
4. SIMULATION RESULTS

In this section, the frequency estimators described earlier were examined on several data sets generated by the equation

$$y(n) = a_1 e^{j(2\pi f_1 n + \phi_1)} + a_2 e^{j(2\pi f_2 n + \phi_2)} + v(n),$$

where $a_1 = 1.0$, $a_2 = 1.0$, $f_1 = 0.5$, $f_2 = 0.52$ and $n = 1, 2, \ldots, N = 25$. The $\phi_i$ are independent random variables uniformly distributed over the interval $[-\pi, \pi]$. The noise $v(k)$ is assumed to be white and uncorrelated with the signal. Note that $f_2 - f_1 < \frac{1}{N}$.

The SNR for either sinusoids is defined as $10 \log_{10} \left( \frac{\sigma_x^2}{\sigma_n^2} \right)$, where $x(n) = a_1 e^{j(2\pi f_1 n + \phi_1)} + a_2 e^{j(2\pi f_2 n + \phi_2)}$ and $\sigma_x^2$, $\sigma_n^2$ are the variances of $x(n)$ and $v(n)$, respectively. The covariance matrix is constructed using forward-backward method to increase robustness. The size of the covariance matrix is chosen to be $p = 10$ which in the absence of noise has effective rank two.

The simulations results of applying the Rational-MUSIC are summarized as follows. A set of 100 random experiments for different SNR=20, 15, 10 dB, dimension of data vectors $p=10$.

<table>
<thead>
<tr>
<th>SNR</th>
<th>$f_1$</th>
<th>$f_2$</th>
<th>RMSE$_{f_1}$</th>
<th>RMSE$_{f_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20 dB</td>
<td>0.500956</td>
<td>0.524952</td>
<td>0.00813</td>
<td>0.019204</td>
</tr>
<tr>
<td>15 dB</td>
<td>0.500729</td>
<td>0.521735</td>
<td>0.00652</td>
<td>0.014531</td>
</tr>
<tr>
<td>10 dB</td>
<td>0.500961</td>
<td>0.524952</td>
<td>0.00813</td>
<td>0.019204</td>
</tr>
</tbody>
</table>

Table 1: Rational-MUSIC: Mean and standard deviation of frequencies for data of two complex sinusoids at frequencies 0.50 and 0.52 in white noise with SNR=20, 15, 10 dB, dimension of data vectors $p=10$.

<table>
<thead>
<tr>
<th>SNR</th>
<th>$f_1$</th>
<th>$f_2$</th>
<th>RMSE$_{f_1}$</th>
<th>RMSE$_{f_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20 dB</td>
<td>0.50022</td>
<td>0.52133</td>
<td>0.00442</td>
<td>0.00514</td>
</tr>
<tr>
<td>15 dB</td>
<td>0.50076</td>
<td>0.52182</td>
<td>0.00781</td>
<td>0.00846</td>
</tr>
<tr>
<td>10 dB</td>
<td>0.50123</td>
<td>0.52214</td>
<td>0.01307</td>
<td>0.01045</td>
</tr>
</tbody>
</table>

Table 2: Standard-MUSIC: Mean and RMSE of frequencies for data of two complex sinusoids at frequencies 0.50 and 0.52 in white noise with SNR=20, 15, 10 dB, dimensions of data vectors and signal subspace are $p=10, q=2$.

For each experiment, (with data length $N=10$ and SNR fixed), we performed 100 independent trials to estimate the frequencies. We used the following performance criterion (RMSE)

$$RMSE = \sqrt{\frac{1}{N_r} \sum_{i=1}^{N_r} (f_i - f_{true})^2}$$

to compare the results. Here $N_r$ is the number of independent realizations, and $f_i$ is the estimate provided from the $i$th realization. The simulations results of applying the Rational-MUSIC are summarized as follows. A set of 100 random experiments for different SNR=20, 15, 10 dB using $\hat{R}_y^m$ with $m=3$ were used. The threshold $b$ in these simulations was estimated by $b = \frac{Tr(\hat{R}_y)}{L}$. The peak spectrum was computed using 1000 frequency bins covering a normalized frequency range of 0 to 1. The mean and RMSE are taken only over realizations where two peaks have occurred. The mean values and the RMSE of the estimated frequencies are given in Table 1. As can be noticed, the performance in these cases are almost identical to those obtained from using the exact decomposition. We repeated the experiment using the standard-MUSIC estimator. The results of testing this algorithm for different SNR were averaged over 100 trials and the mean and RMSE of each frequency was presented in Table 2. Figure 1 shows the peaks resulted from applying Rational-MUSIC for 30 independent trials with SNR=15 dB and using $\hat{R}_y(10)^m$ for $m=3$. The corresponding results obtained using the standard MUSIC are shown in Figure 2. As can be noticed from Tables 1-2 and Figures 1-2, both Rational-MUSIC and the standard MUSIC are seen to have virtually identical performance.

When $m$ is small, it is observed that overestimation of $q$ leads to better estimation of the frequencies. The robustness of the methods against overestimation of the number of sources $q$ can be explained as follows. Overestimation of $q$ means additional vectors are included in the basis of the signal subspace. In our approximation, these vectors are not purely noise but contain some signal component. In the standard MUSIC, if additional vector is added to the basis, a purely noise vector is included in the signal subspace causing spurious peaks. Note that when $m$ is small the first few signal vectors contain noise components since the noise and signal vectors are not well-separated. Thus we have to consider more vectors to capture the signal subspace. However, this separation becomes more prominent as $m$ increases.

5. CONCLUSION

Eigendecomposition-based methods such as MUSIC and Minimum Norm, estimators are popular for their high resolution property in sinusoidal and direction of arrival estimation but they are also known to be of high computational demand. In this paper, new
Fast and robust algorithms for DOA and sinusoidal frequency estimation are presented. These algorithms approximate the required subspace using rational and power-like methods applied to the sample covariance matrix. The operation count of these algorithms were shown to be much less than that associated with the eigendecomposition-based methods. Furthermore, this approach can be refined to compute invariant subspaces of any complex matrix in different regions in the complex plane.

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**References**


*Figure 1: Spectral peaks at \( f_1 = 0.5 \) and \( f_2 = 0.52 \) with \( SNR = 15dB \), resulted from applying the Rational-MUSIC, with \( m = 3 \) for 30 independent trials.*

*Figure 2: Spectral peaks at \( f_1 = 0.5 \) and \( f_2 = 0.52 \) with \( SNR = 15dB \), resulted from applying SVD-based MUSIC on 30 independent trials with \( q = 2 \) and \( p = 10 \).*