NEW TIME-FREQUENCY SYMBOL CLASSIFICATION

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ABSTRACT

We propose new time-frequency (TF) symbols as the narrowband Weyl symbol (WS) smoothed by an appropriate kernel. These new symbols preserve time and frequency shifts on a random process. Choosing specific smoothing kernels, we can obtain various new symbols (e.g., Levin symbol and Page symbol). We link a quadratic form of the signal to the new symbols and Cohen’s class of quadratic time-frequency representations, and we derive a simple kernel constraint for unitary symbols. We also propose an affine class of symbols in terms of the wideband Weyl symbol ($P_0$-WS). These symbols preserve scale changes and time shifts. Furthermore, we generalize the smoothed versions of the WS and $P_0$-WS to analyze random processes undergoing generalized frequency shifts or generalized time shifts.

1. INTRODUCTION

The narrowband Weyl symbol (WS) has been successfully used in the time-frequency (TF) analysis of linear time-varying systems and nonstationary processes [4, 10, 15]. The WS of a linear operator $\mathcal{L}$ is defined as

$$WS_{\mathcal{L}}(t, f) = \int K_\mathcal{L}(t + \tau/2, t - \tau/2) e^{j2\pi f \tau} d\tau,$$

where $K_\mathcal{L}(t, \tau) = \sum_{n} \mu_n u_n^*(\tau) u_n(t)$ is the kernel of the operator $\mathcal{L}$ defined on $L_2(\mathbb{R})$ [5], $\mu_n$ and $u_n(t)$ are the eigenvalues and eigenfunctions, respectively, of $K_\mathcal{L}(t, \tau)$, and $W_{\mathcal{L}}$ is the Wigner distribution (WD) of $u_n(t)$ [6]. Based on (2), the quadratic time-frequency representation (QTFR). The WS can be interpreted as a time-varying spectrum of a random process. It is a useful TF analysis tool as it preserves constant time shifts and frequency shifts on a random process $x(t)$ [4],

$$y(t) = x(t-\tau) \Rightarrow WS_{\mathcal{L}}(t, f) = WS_{\mathcal{L}}(t-\tau, f),$$

$$y(t) = x(t)e^{j2\pi f \tau} \Rightarrow WS_{\mathcal{L}}(t, f) = WS_{\mathcal{L}}(t, f-\nu),$$

where $R_x$ is the autocorrelation operator of $x(t)$. The WS also preserves scale changes on the random process [12],

$$y(t) = \sqrt{\alpha} x(at) \Rightarrow WS_{\mathcal{L}}(t, f) = WS_{\mathcal{L}}(at, f/a).$$

The Weyl correspondence is a unitary mapping between the operator $\mathcal{L}$ and its WS [14, 4, 10, 15]. In particular, the WS (and its 2-D Fourier transform (FT), the spreading function, $SF_{\mathcal{L}}(\tau, \nu)$) provide an important formulation of a quadratic form of the process $x(t)$,

$$\int (Lx)(t) z^*(dt) dt = \int \int WS_{\mathcal{L}}(t, f) W_{\mathcal{L}}(t, f) dtdf,$$

where $AF_\nu(\tau, \nu)$ is the ambiguity function (AF) of $x(t)$ [6]. This important relationship in (6) is useful for a TF concentration measure definition [15] and for TF detection applications [11].

The wideband Weyl symbol ($P_0$-WS) in [16, 7, 8] is defined as

$$P_0WS_{\mathcal{B}}(t, f) = \int TS_{\mathcal{B}}(\lambda(\nu)e^{j2\pi f \tau} \lambda(\nu)e^{-j2\pi f \tau}) d\lambda,$$

where $TS_{\mathcal{B}}(f, \nu) = \sum_{n} \beta_n P_{\mathcal{B}n}(t, f), f > 0$

and $P_{\mathcal{B}n}(t, f)$ is the kernel of the operator $\mathcal{B}$ on $L_2(\mathbb{R}^+)$, $\beta_n$ and $Q_{\mathcal{B}n}(f)$ are the eigenvalues and eigenfunctions, respectively, of $\mathcal{B}$. In (8), the $P_0$-WS is associated with the unitary Bertrand $P_0$-distribution of $Q_{\mathcal{B}n}(f)$ [1], and $A(\nu) = e^{-j\nu/2}$$. In (8), the unitary $P_0$-WS is associated with the unitary Bertrand $P_0$-distribution. The $P_0$-WS is an important operator symbol that satisfies the time-shift covariance in (3) and scale covariance in (5). It also satisfies hyperbolic time-shift covariance [8]. A quadratic form of the process $x(t)$ with FT $X(f)$ can be expressed in the frequency domain in terms of the $P_0$-WS and the unitary Bertrand $P_0$-distribution

$$\int_{0}^{\infty} (BX)(f) X^*(f) df = \int_{0}^{\infty} P_0WS_B(t, f) P_0x(t, f) dt df.$$

In [14], it was noted that if one chooses a different QTFR than the WD in (6), then one can have an alternative mapping between a linear operator and the symbol corresponding to the QTFR. In this paper, we propose a class of operator symbols to (i) satisfy the desirable TF shift covariance properties (3) and (4), and (ii) provide alternative formulations of the quadratic form in (6) using corresponding QTFRs in Cohen’s class [2, 6]. We express this new class$^3$ of TF symbols in terms of a kernel function$^2$ that is related to the kernel of a Cohen’s class QTFR. If the associated Cohen’s class QTFR is unitary, then the symbol kernel satisfies a simple constraint for unitarity. We show that the conventional WS in (1) is a member of this class, and that all other members can be written as smoothed versions of the WS. Other members include the Kohn-Nirenberg symbol [4], the $\alpha$-generalized WS [10], as well as the new Levin symbol, the new Page symbol, and the new pseudo WS proposed in this paper.

We also propose new operator symbols based on an affine class kernel formulation; they satisfy the time-shift covariance property in (3) and the scale covariance property in (5). This affine class of symbols can be written both in terms of the WS in (1) and the $P_0$-WS in (7). We derive the symbol kernel constraint necessary for these new affine symbols to provide alternative formulations of the quadratic form in (9). Specific members of this class of symbols include the conventional WS in (1), the $P_0$-WS in (7), and new

$^2$This paper, by the term “QTFR class” ("TF symbol class"), we mean the group of all QTFRs (TF symbols) that satisfy a given set of properties. Also, by the term “kernel”, we mean the function that uniquely characterizes a specific member of a group. Here, we talk about three different types of kernels: (a) kernel of an operator (cf. $K_\mathcal{L}$ in (1)), (b) kernel of a symbol (cf. $A_\nu(\tau, \nu)$ in (10)), and (c) kernel of a QTFR (cf. $A_\nu(\tau, \nu)$ in (13)).

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symbols corresponding to the unitary Bertrand $P_\tau$-distributions [1]. Finally, we extend the symbol class formulation concept to new hyperbolic symbols preserving scale changes and hyperbolic frequency shifts, to new power symbols preserving scale changes and power time shifts, to new exponential symbols preserving frequency shifts and exponential time shifts, and to new generalized warped symbols preserving generalized time shifts or generalized frequency shifts.

2. CLASS OF TF SHIFT COVARIANT SYMBOLS

We propose a new class of operator symbols which preserve time and frequency shifts (like the WS in (3) and (4)), an important property for the TF analysis of random processes. We define these symbols as smoothed versions of the conventional WS in (1) [9]

$$S_{\tau}(t, f) = \int \theta_T(\tau, \nu) \Psi_T(\tau, \nu) e^{-j2\pi(\nu t - \tau f)} d\tau d\nu$$

where $\theta_T(\tau, \nu) = 2-D$ FT$\{\Psi_T(\tau, \nu)\}$ is a 2-D kernel function that uniquely characterizes the operator symbol, $T_{\tau}(t, f)$. The new symbol class provides a new quadratic form to (6).

The QFT symbol is defined in (12) as a member of Cohen’s class of TF shift covariant QFTs [2, 6]

$$Q_{\tau}(t, f) = \int \Psi_T(\tau, \nu) A_T(\tau, \nu) e^{-j2\pi(\nu t - \tau f + \tau)} d\tau d\nu$$

where $\Psi_T(\tau, \nu)$ is a 2-D kernel that uniquely characterizes the QFT, $T_{\tau}(t, f)$. For (12) to hold, the QFT symbol is defined in (13) as a kernel $\Psi_T(\tau, \nu)$ that is related to the symbol kernel $\Psi_T(\tau, \nu)$ in (11) as

$$\Psi_T(\tau, \nu) = 1 / \Psi_T(\tau, \nu) = \Psi_T(\tau, \nu) / \Psi_T(\tau, \nu)$$

For example, in the WS in (1) is a member of the symbol class (10)-(11) when the symbol kernel $\Psi_{WS}(\tau, \nu)$ is 1. The Cohen’s class QFT $T^{C}(\tau, \nu)$ in the quadratic form (12) must be the QFT with kernel $\Psi^{C}(\tau, \nu) = 1 / \Psi_{WS}(\tau, \nu) = 1$, which corresponds to the WD (cf. quadratic form in (6)). Note that the quadratic form in (12) can also be written as

$$\langle L, x \rangle (t, f) = \int \int \theta_T(\tau, \nu) d\tau d\nu$$

where $\theta_T(\tau, \nu) = \Psi_T(\tau, \nu) A_T(\tau, \nu) = 2-D$ FT$\{\Psi_T(\tau, f)\}$ and $T_{\tau}(t, f) = 2-D$ FT$\{T_{\tau}(t, f)\}$.

2.1. Unitary TF Shift Covariant Symbols

The formulation in (10) provides a new mapping between a linear operator $L$ and its new symbol, $T_{\tau}(t, f)$. We can show that this mapping is unitary if the symbol is unitary, i.e. if the symbol satisfies the relation in (9)

$$\int \int S_{\tau}(t, f) T_{\tau}(t, f) d\tau d\nu = \sum \mu_n \chi_n(\tau) \gamma_n(\nu)$$

where $\mu_n$ and $\nu_n (\nu)$ are the eigenvalues and eigenfunctions, respectively, of the kernel of the operator $L (\nu)$. We can further show that (15) is satisfied if and only if the symbol kernel $\Psi_T(\tau, \nu)$ in (11) satisfies the constraint $\Psi_T(\tau, \nu) = 1$. The associated Cohen’s class QFT $T^{C}(\tau, \nu)$ in (12) is also unitary [6] and its kernel satisfies $\Psi_T(\tau, \nu) = 1$. Thus, we say that this unitary symbol is associated with the unitary QFT. Let us denote the unitary symbol as $U^{T}(t, f)$ and its associated unitary QFT in (12) as $U^{T}(t, f)$. Using the relation in (14) and the unitary QFT constraint $\Psi_T(\tau, \nu) = 1$, we can show that the unitary symbol kernel equals the unitary Cohen’s class QFT kernel, i.e. $\Psi_T(\tau, \nu) = \Psi_T(\tau, \nu)$. This leads to the useful relation

$$U^{T}(t, f) = \sum \mu_n \Psi_T(\tau, \nu)$$

between the unitary symbol satisfying (15) and its associated QFT (cf. 2) where $\mu_n$ and $\nu_n (\nu)$ are the eigenvalues and eigenfunctions of the kernel of $L$ in (1)-(2).

2.2. Examples of Unitary TF Shift Covariant Symbols

Some examples of unitary symbols in (10)-(11) satisfying the constraint $\Psi_T(\tau, \nu) = 1$ are summarized in Table 1 below:

- Narrowband Weyl symbol: $W_{\tau}(t, f)$ is defined in (1) with kernel $\Psi_{WS}(\tau, \nu) = 1$ or $\Psi_{WS}(t, f) = \delta(t) \delta(f)$.
- Kohn-Nirenberg symbol: The Kohn-Nirenberg symbol of an operator associated with the Rihaczek distribution is defined as [4]

$$\Psi_{K}(\tau, \nu) = e^{-j2\pi \nu t}$$

where $\Psi_{K}(\tau, \nu) = e^{-j2\pi \nu t}$. The Kohn-Nirenberg symbol is also known as the Zadeh’s transfer function [10].
- Levin symbol: We define the symbol associated with the Levin distribution as

$$L_{\tau}(t, f) = \int e^{-j2\pi \nu t} d\nu$$

The smoothing kernel for the Levin symbol is $\Psi_{\tau}(\tau, \nu) = e^{-j2\pi \nu t}$. The Levin symbol is also known as the Zadeh’s transfer function [10].
- Page symbol: We define the new symbol, $P_{\tau}(t, f)$, associated with the Page distribution as

$$P_{\tau}(t, f) = \int e^{-j2\pi \nu t} d\nu$$

with smoothing kernel $\Psi_{\tau}(\tau, \nu) = e^{-j2\pi \nu t}$. The Page symbol is also known as the Zadeh’s transfer function [10].
- $\alpha$ Generalized WS: The symbol associated with the $\alpha$-generalized WD [6] is defined in (10, 12) as

$$W_{\alpha}(t, f) = \int e^{(t - \tau) + \alpha} \mid \tau - \frac{1}{2} \alpha \mid d\tau$$

Its smoothing kernel is $\Psi_{\alpha}(\tau, \nu) = e^{-j2\pi \nu t}$. Note that when $\alpha = 0$, $W_{\alpha}(t, f)$ reduces to the conventional WS in (1), and when $\alpha = 1/2$, it simplifies to the Kohn-Nirenberg symbol in (17).

Table 1: Various unitary smoothed versions of the narrowband WS in (11). $K_{\tau}(\tau, \nu)$ is the kernel of the operator $L$ defined on $L_2(\mathbb{R})$.

<table>
<thead>
<tr>
<th>Symbol Name</th>
<th>Unitary Time-Frequency Symbols</th>
<th>Smoothing Kernel, $\Psi_T(\tau, \nu)$</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Narrowband WS</td>
<td>$W_{\tau}(t, f) = \int e^{-j2\pi \nu t} d\nu$</td>
<td>1</td>
<td>[4, 10, 15]</td>
</tr>
<tr>
<td>Kohn-Nirenberg symbol</td>
<td>$K_{\tau}(\tau, \nu) = e^{-j2\pi \nu t}$</td>
<td>exp$(-j2\pi \nu t)$</td>
<td>[4]</td>
</tr>
<tr>
<td>Levin symbol</td>
<td>$L_{\tau}(t, f) = \int e^{-j2\pi \nu t} d\nu$</td>
<td>exp$(-j2\pi \nu t)$</td>
<td>new</td>
</tr>
<tr>
<td>Page symbol</td>
<td>$P_{\tau}(t, f) = \int e^{-j2\pi \nu t} d\nu$</td>
<td>exp$(-j2\pi \nu t)$</td>
<td>new</td>
</tr>
<tr>
<td>$\alpha$ Generalized WS</td>
<td>$W_{\alpha}(t, f) = \int e^{(t - \tau) + \alpha} \mid \tau - \frac{1}{2} \alpha \mid d\tau$</td>
<td>exp$(-j2\pi \nu t)$</td>
<td>[10, 12]</td>
</tr>
</tbody>
</table>
2.3. Example of Non-Unitary TF Shift Covariant Symbol

When \( \Theta_{TS}^{(C)}(\tau, \nu) \neq 1 \), the symbol TS\(^{(C)}\) in (10)-(11) is not unitary. As a consequence, the symbol does not satisfy the unitary relation in (15) and cannot be written as in (16). However, the symbol still satisfies the covariance properties in (3)-(4) and provides an alternative quadratic form of a random process in (12). An example of a non-unitary symbol follows:

**Pseudo WS:** We define the new pseudo WS that corresponds in (12) to the pseudo WD with zero window \( \eta(t) \) as

\[
\text{PWS}_\zeta(t, f) = \int \frac{1}{\eta(t)} K(t, f) dt \frac{e^{-j2\pi ft}}{2}\, dt.
\]

Since the kernel of the pseudo WD is \( \Psi_{\text{PWS}}^{(C)}(\tau, \nu) = \eta(t)\eta(-t) \)

in (13), then, using (14), the corresponding symbol kernel of the PWS in (11) is \( \Theta_{\text{PWS}}^{(C)}(\tau, \nu) = 1/\eta(t) \eta(-t) \).

3. CLASS OF AFFINE TF SYMBOLS

The class of symbols in (10) does not necessarily preserve scale changes on a random process as in (5). Thus, we propose a new class of TF symbols covariant to time shifts in (3) and scale changes in (5) of a random process. We denote these new symbols with the superscript (A) for affine. We define them as affine smoothed versions of the WS in (1) or of the P0WS in (7).

\[
T_{SA}^{(A)}(t, f) = \int_{0}^{\infty} \theta_{SA}^{(A)}(t, f) dt \frac{e^{-j2\pi ft}}{2}\, df,
\]

where \( \theta_{SA}^{(A)}(t, f) = \int \theta_{SA}^{(A)}(t, f) \frac{e^{-j2\pi ft}}{2}\, df \)

is a 2-D kernel characterizing the symbol \( \Theta_{SA}^{(A)}(\zeta, \beta) = \mathcal{F}^{-1}\mathcal{B}\mathcal{F} \), the FT operator \( \mathcal{F} \) is \( (\mathcal{F}x)(f) = \int x(t) e^{-j2\pi ft} dt \).

We can show that if the affine symbol in (19) is unitary, i.e., satisfies (15), then its corresponding kernel satisfies the constraint

\[
\int_{\mathbb{R}^2} \theta_{SA}^{(A)}(c, b) \theta_{SA}^{(A)}(\zeta, \beta) e^{j2\pi c\zeta + j2\pi b\beta} d\zeta d\beta = \delta(b - 1), \quad \forall \zeta.
\]

The unitary affine symbols denoted UT\(T_{SA}^{(A)}(t, f) \) provide a new formulation of the quadratic form in (9),

\[
\int_{0}^{\infty} \langle X(t, f) \rangle^2 dt = \int_{0}^{\infty} \langle \text{UT}_{SA}^{(A)}(t, f) \rangle \langle \text{UT}_{SA}^{(A)}(t, f) \rangle dt df.
\]

3.1. Examples of Unitary Affine Symbols

**Narrowband Weyl symbol:** When the kernel symbol \( \theta_{SA}^{(A)}(c, b) = \delta(c)\delta(b + 1) \) in (19), the conventional WS in (1) is obtained.

**P0-Weyl symbol:** \( T_{SA}^{(A)}(t, f) \) in (20) simplifies to the P0WS in (7) when the kernel \( \theta_{SA}^{(A)}(c, b) = \delta(c)\delta(b + 1) \).

**\( \alpha \)-Generalized Weyl symbol:** The \( \alpha \)-generalized WS in (18) is obtained from (19) when \( \theta_{SA}^{(A)}(c, b) = e^{-j2\pi \alpha (b+1)/\alpha} \).

**Bertrand P\( \alpha \)-symbols:** We propose the new symbols associated with the unitary Bertrand P\( \alpha \)-distributions [1] and define them as

\[
P_{P\alpha}^{(A)}(t, f) = \int \theta_{SA}^{(A)}(t, f, \lambda_{\alpha}(c), \lambda_{\alpha}(\alpha)) e^{-j2\pi \alpha (b+1)/\alpha} \mu(c) dc, \quad \kappa \neq 0, 1
\]

where \( \lambda_{\alpha}(c) = [e^{c\alpha} - 1] / (e^{c\alpha} - 1) \) and \( \mu(c) = (\lambda_{\alpha}(c) - \lambda_{\alpha}(\alpha))^{1/2} \).

4. OTHER CLASSES OF TF SYMBOLS

4.1. Hyperbolic Smoothed Symbols

We define the class of hyperbolic (note the superscript (H)) smoothed symbols for \( t > 0 \) as

\[
\text{HTS}_{SA}^{(H)}(t, f) = \int_{0}^{\infty} \theta_{SA}^{(H)}(t, f, \lambda_{H}(c), \lambda_{H}(\alpha)) e^{-j2\pi \alpha (b+1)/\alpha} \mu(c) dc, \quad \kappa \neq 0, 1
\]

where the operator \( \mathcal{H} \) is defined on \( L_2(\mathbb{R}^+) \) and \( \theta_{SA}^{(H)}(c, b) = \theta_{SA}^{(A)}(c, b) \).

4.2. Power smoothed symbols

We define the new power class of TF symbols for \( f > 0 \) as

\[
\text{PTS}_{SA}^{(P)}(t, f) = \int_{0}^{\infty} \theta_{SA}^{(P)}(t, f, \lambda_{P}(c), \lambda_{P}(\alpha)) e^{-j2\pi \alpha (b+1)/\alpha} \mu(c) dc, \quad \kappa \neq 0, 1
\]

where \( \theta_{SA}^{(P)}(c, b) = e^{-j2\pi \alpha (b+1)/\alpha} \).

Note that this symbol reduces to the affine smoothed symbols in (20).
where $f_r > 0$ is a fixed reference frequency.

4.3. Exponential smoothed symbols

We define the new class of exponential smoothed symbols as

$$y(t) = \sqrt{a_1 x(at)} \Rightarrow \text{PTS}_{R_0}(t, f) = \text{PTS}_{R_0}(at, f/a)$$

where $f_r > 0$ is a fixed reference frequency.

4.4. Generalized frequency-shift covariant symbols

We propose a new class of TF symbols, GTS$^{(GC)}(t, f)$, that preserves general frequency shifts on a random process. The shift depends on a one-to-one warping function $\xi(b)$

$$y(t) = x(t)e^{2\pi i \xi(t) \xi'(t) / \xi(t)} \Rightarrow \text{GTS}_{R_0}(t, f) = \text{GTS}_{R_0}(t - \xi(t) / \xi(t)), f)$$

where $\xi(b) = \frac{d}{db} \xi(b)$. The new symbols are defined as

$$\text{GTS}_{R_0}^{(GC)}(t, f) = \int \text{GTS}^{(GC)}_{R_0}(\xi(t), f) = \int \text{GTS}_{R_0}^{(GC)}(t - \xi(t) / \xi(t)), f)$$

$$= \text{GWS}_{R_0}(t, f)$$

$$= \text{TS}_{R_0}^{(GC)}(t - \xi(t) / \xi(t)), f)$$

where $\text{GWS}_{R_0}$ is defined on $L_2([p, q])$, and $[p, q]$ is determined by the domain of the warping function $\xi(b)$, $\text{GTS}_{R_0}^{(GC)}(t, f)$ is defined in (10), the warping is $W_2(x(t)) = \text{ex}(t, \xi(t), f) = (\xi(t) / \xi(t))^{1/2}$, and $W_2 x(t)$ is $x(t)$.

Depend on the smoothing kernel $\omega_{\text{TS}}(c, b)$ in (26), GTS$^{(GC)}_{R_0}$ results in the new generalized Levin symbols, the generalized Page symbol or the generalized Kohn-Nirenberg symbol [9]. Also, we can derive special cases of GTS$^{(GC)}_{R_0}$ by choosing a specific warping function $\xi(b)$ in (26). For example, we obtain the hyperbolic smoothed symbols in Section 4.1 when $\xi(b) = In(b)$.

4.5. Generalized time-shift covariant symbols

We define a class of generalized time-shift covariant symbols as

$$\text{GTS}^{(GA)}(t, f) = \int \text{GTS}^{(A)}(t - \xi(t) / \xi(t), f)$$

$$\Rightarrow \text{GTS}_{R_0}^{(GA)}(t, f) = \text{GTS}_{R_0}^{(GA)}(t - \xi(t) / \xi(t), f)$$

where $\text{GTS}^{(A)}(c, b) = \text{TS}_{R_0}(c, b)$ in (20) and GWS$^{(A)}_{R_0}(t, f)$ is the generalized $\text{GWS}_{R_0}$ [8]. The GTS$^{(GA)}_{R_0}$ preserves generalized time shifts on a random process $X(f)$. 

$$y(t) = \left(e^{-j2\pi \text{Re}(X(f))} X(f)\right)$$

Depending on the warping function $\xi(b)$, the GTS$^{(GA)}_{R_0}$ in (27) can be simplified to a specific class of affine symbols. For example, we obtain the TS$^{(A)}_{R_0}$ in (20) when $\xi(b) = b$, the PTS$^{(A)}_{R_0}$ in (24) when $\xi(b) = b^*$ and the ETS$^{(A)}_{R_0}$ in (25) when $\xi(b) = b^*$. 

5. CONCLUSION

In this paper, we proposed new classes of smoothed versions of the narrowband Weyl symbol and the broadband Weyl symbol. These new symbols preserve important changes on a random process. For example, a smoothed version of the broadband WS preserves constant TF shifts on a random process and belongs to the new class of TF shift covariant symbols. The new symbols formulate the quadratic form of a random process with corresponding QTFRs. We showed that a symbol kernel is identical to its associated QTFR kernel if the QTFR is unitary. We also proposed generalized formulations of smoothed symbols. We provided special examples of generalized smoothed symbols (e.g., hyperbolic smoothed symbols, power smoothed symbols, and exponential nonsmoothed symbols). For example, we derived the hyperbolic Weyl symbol, hyperbolic Kohn-Nirenberg symbol, hyperbolic Levin symbol and hyperbolic Page symbol as hyperbolic smoothed symbols.

6. REFERENCES


